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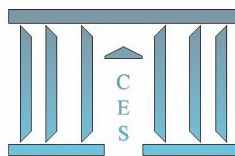
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**Price revelation and existence of equilibrium
in a private belief economy**

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PRICE REVELATION AND EXISTENCE OF EQUILIBRIUM
IN A PRIVATE BELIEF ECONOMY

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Abstract

We consider a pure exchange financial economy, where rational agents, possibly asymmetrically informed, forecast prices privately and, therefore, face ‘exogenous uncertainty’, on the future state of nature, and ‘endogenous uncertainty’, on future prices. At a sequential equilibrium, all agents expect the ‘true’ price as a possible outcome and elect optimal strategies at the first period, which clear on all markets ex post. We introduce no-arbitrage prices and display their revealing properties. Under mild conditions, we show that a sequential equilibrium exists, whatever the financial structure and agents’ private information or beliefs. This result suggests that existence problems of standard sequential models, following Hart (1975) or Radner (1979), stem from the rational expectation and perfect foresight assumptions.

Key words: sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

JEL Classification: D52

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1 Introduction

The traditional approach to sequential financial equilibrium relies on Radner's (1972-1979) classical, but restrictive, assumptions that agents have the so-called 'rational expectations' of private information signals, and 'perfect foresight' of future prices. Along the former assumption, agents are endowed, quoting Radner, with 'a model' of how equilibrium prices are determined and (possibly) infer private information of other agents from comparing actual prices and price expectations with theoretical values at a price revealing equilibrium. Along the latter assumption, agents anticipate with certainty exactly one price for each commodity (or asset) in each prospective state, which turns out to be the true price if that state prevails. Both assumptions presume much of agents' inference capacities. Both assumptions lead to classical cases of inexistence of equilibrium, as shown by Radner (1979), Hart (1975), Momi (2000), Busch-Govindan (2004), among others. Building on our earlier papers, we argue hereafter that the relevance and properties of the sequential equilibrium model can jointly be improved, if we drop these standard assumptions.

In a first model [4], dropping rational expectations only, we provided the basic tools, concepts and properties for an arbitrage theory, embedding jointly the symmetric and asymmetric information settings. In this model, we showed in [6], standard existence problems of asymmetric information vanished, namely, a financial equilibrium with nominal assets existed, not only generically - as in Radner's (1979) rational expectations model - but under the very same no-arbitrage condition, with symmetric or asymmetric information, namely, under the generalized no-arbitrage condition introduced in [4]. This result was consistent and extended David Cass' (1984) standard existence theorem to the asymmetric information setting.

We now drop both assumptions of rational expectations and perfect foresight, letting agents form their anticipations privately. Thus, rational agents can no longer be certain which price might prevail, as in the classical model. Equilibrium prices now depend on all agents' forecasts today, which are private knowledge. So, agents also face uncertainty about future prices. To encapture this, we introduce a two-period pure exchange economy, where agents, possibly asymmetrically informed, face exogenous uncertainty, represented by finitely many states of nature, exchange consumption goods on spot markets, and - nominal or real - assets on financial markets, but also face an '*endogenous uncertainty*' on future prices, in each state they expect. Agents now have private sets of price forecasts, distributed along idiosyncratic probability laws, called beliefs. We refer to the latter uncertainty as endogenous because it both affects and is attached to the endogenous price variables.

The current model's equilibrium, or '*correct foresight equilibrium*' (C.F.E.), is reached when all agents, today, anticipate tomorrow's '*true*' price as a possible outcome, and elect optimal strategies, which clear on all markets at every time-period. This equilibrium concept is, indeed, a *sequential* one. It differs from the traditional *temporary* equilibrium notion, introduced by Hicks (1939) and developed, later, by Grandmont (1977, 1982), Green (1973), Hammond (1983), Balasko (2003), among others. With temporary equilibrium, anticipations may also be set exogenously, but can lead agents, ex post, to revise their plans and beliefs, face bankruptcy or welfare increasing retrade opportunities. Such outcomes are inconsistent with our model.

After presenting the model, we introduce a notion of no-arbitrage prices, which always exist, encompass equilibrium prices, and may reveal information to agents having no clue of how equilibrium prices are determined. We show that any agent can infer enough information from such prices to free markets from arbitrage.

Next, we study the existence issue, and suggest the correct foresight equilibrium may solve the classical existence problems, which followed, not only Radner's (1979) rational expectations equilibrium (as we had already shown in [6]), but also Hart (1975), Momi (2001), Busch-Govindan (2004), among others. We prove hereafter that a C.F.E. exists whenever agents' anticipations embed a so-called '*minimum uncertainty set*', corresponding to the incompressible uncertainty which may remain in a private belief economy. Thus, equilibrium prices always exist, and reveal to rational agents, whenever required, their own sets of anticipations at equilibrium.

The paper is organized as follows: we present the model, in Section 2, the concept of no-arbitrage prices and the information they reveal, in Section 3, the minimum uncertainty set and the existence Theorem, in Section 4. We prove this theorem, in Section 5, differing to an Appendix the proof of technical Lemmas.

2 The basic model

We consider a pure-exchange economy with two periods ($t \in \{0, 1\}$), a commodity market and a financial market, where agents (at $t = 0$) may be asymmetrically informed and face an endogenous uncertainty on future prices. The sets of agents, $I := \{1, \dots, m\}$, commodities, $\mathcal{L} := \{1, \dots, L\}$, states of nature, $S := \{1, \dots, N\}$, and assets, $\mathcal{J} := \{1, \dots, J\}$, are all finite (i.e., $(m, L, N, J) \in \mathbb{N}^4$).

2.1 The model's notations

Throughout, we denote by \cdot the scalar product and $\|\cdot\|$ the Euclidean norm on an Euclidean space and by $\mathcal{B}(K)$ the Borel sigma-algebra of a topological space, K . We let $s = 0$ be the non-random state at $t = 0$ and $S' := \{0\} \cup S$. For all set $\Sigma \subset S'$ and tuple $(s, l, x, x', y, y') \in \Sigma \times \mathcal{L} \times \mathbb{R}^\Sigma \times \mathbb{R}^\Sigma \times \mathbb{R}^{L\Sigma} \times \mathbb{R}^{L\Sigma}$, we denote by:

- $x_s \in \mathbb{R}$, $y_s \in \mathbb{R}^L$ the scalar and vector, indexed by $s \in \Sigma$, of x , y , respectively;
- y_s^l the l^{th} component of $y_s \in \mathbb{R}^L$;
- $x \leq x'$ and $y \leq y'$ (respectively, $x << x'$ and $y << y'$) the relations $x_s \leq x'_s$ and $y_s^l \leq y_s'^l$ (resp., $x_s < x'_s$ and $y_s^l < y_s'^l$) for each $(l, s) \in \{1, \dots, L\} \times \Sigma$;
- $x < x'$ (resp., $y < y'$) the joint relations $x \leq x'$, $x \neq x'$ (resp., $y \leq y'$, $y \neq y'$);
- $\mathbb{R}_+^{L\Sigma} = \{x \in \mathbb{R}^{L\Sigma} : x \geq 0\}$ and $\mathbb{R}_+^\Sigma := \{x \in \mathbb{R}^\Sigma : x \geq 0\}$,
 $\mathbb{R}_{++}^{L\Sigma} := \{x \in \mathbb{R}^{L\Sigma} : x >> 0\}$ and $\mathbb{R}_{++}^\Sigma := \{x \in \mathbb{R}^\Sigma : x >> 0\}$;
- $\mathcal{M}_0 := \{(p_0, q) \in \mathbb{R}_+^L \times \mathbb{R}^J : \|p_0\| + \|q\| = 1\}$;
- $\mathcal{M}_s := \{(s, p) \in S \times \mathbb{R}_+^L : \|p\| = 1\}$, for every $s \in S$;
- $\mathcal{M} := \cup_{s \in S} \mathcal{M}_s$, a topological subset of the Euclidean space \mathbb{R}^{L+1} ;
- $B(\omega, \varepsilon) := \{\omega' \in \mathcal{M} : \|\omega' - \omega\| < \varepsilon\}$, for every pair $(\omega, \varepsilon) \in \mathcal{M} \times \mathbb{R}_{++}$;
- $P(\pi) := \{\omega \in \mathcal{M} : \pi(B(\omega, \varepsilon)) > \varepsilon, \forall \varepsilon > 0\}$, the support of a probability, π , on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$;
- $\pi(P)$, for any closed set, $P \subset \mathcal{M}$, the set of probabilities on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, whose support (as defined above) is P .

2.2 The commodity and asset markets

The L consumption goods, $l \in \mathcal{L}$, may be exchanged by consumers, on the spot markets of both periods. In each state, $s \in S$, an expectation of a spot price, $p \in \mathbb{R}_+^L$, or the spot price, p , in state s itself, are denoted by the pair $\omega_s := (s, p) \in S \times \mathbb{R}_+^L$, and normalized, at little cost, to the above set \mathcal{M}_s .

Each agent, $i \in I$, is granted an endowment, $e_i := (e_{is}) \in \mathbb{R}_+^{LS'}$, which secures her the commodity bundle, $e_{i0} \in \mathbb{R}_+^L$ at $t = 0$, and $e_{is} \in \mathbb{R}_+^L$, in each state $s \in S$, if this

state prevails at $t = 1$. To harmonize notations, we will also denote $e_{i\omega} := e_{is}$, for every triple $(i, s, \omega) \in I \times S' \times \mathcal{M}_s$. Ex post, the generic i^{th} agent's welfare is measured by a continuous utility index, $u_i : \mathbb{R}_+^{2L} \rightarrow \mathbb{R}_+$, over her consumptions at both dates.

The financial market permits limited transfers across periods and states, via J assets, or securities, $j \in \mathcal{J} := \{1, \dots, J\}$, which are exchanged at $t = 0$ and pay off, in commodities and/or in units of account, at $t = 1$. For any spot price, or expectation, $\omega \in \mathcal{M}$, the cash payoffs, $v_j(\omega) \in \mathbb{R}$, of all assets, $j \in \{1, \dots, J\}$, conditional on the occurrence of price ω , define a row vector, $V(\omega) = (v_j(\omega)) \in \mathbb{R}^J$. By definition and from the continuity of the scalar product, the mapping $\omega \in \mathcal{M} \mapsto V(\omega)$ is continuous. The financial structure may be incomplete, namely, the span, $\langle (V(\omega_s))_{s \in S} \rangle := \{(V(\omega_s) \cdot z)_{s \in S} : z \in \mathbb{R}^J\}$ may have lower rank (for all prices $(\omega_s) \in \Pi_{s \in S} \mathcal{M}_s$) than $N := \#S$. From above, assets provide no insurance against endogenous uncertainty.

Agents can take unrestrained positions (positive, if purchased; negative, if sold), in each security, which are the components of a portfolio, $z \in \mathbb{R}^J$. Given an asset price, $q \in \mathbb{R}^J$, a portfolio, $z \in \mathbb{R}^J$, is thus a contract, which costs $q \cdot z$ units of account at $t = 0$, and promises to pay $V(\omega) \cdot z$ units tomorrow, for each spot price $\omega \in \mathcal{M}$, if ω obtains. Similarly, we normalize first period prices, $\omega_0 := (p_0, q)$, to the set \mathcal{M}_0 .

2.3 Information and beliefs

Ex ante, the generic agent, $i \in I$, is endowed with a private idiosyncratic set of anticipations, $P_i \subset \mathcal{M}$, according to which she believes tomorrow's true price (i.e., that which will prevail at $t = 1$) will fall into P_i . Consistently with [4], this set, $P_i \subset S \times \mathbb{R}^L$, encompasses a private information signal, $S_i \subset S$, that the true state will be in S_i . Agents' are assumed to receive no wrong signal, in the sense that no state will effectively prevail tomorrow, out of the pooled information set,

$\underline{S} := \cap_i S_i$. Typically, rational agents' forecasts of the spot prices cannot degenerate to singletons, since all prices depend on other agents' forecasts, which are private. Yet, from observing markets (at $t = 0$), agents may update their beliefs before trading. Resuming the notations of sub-Section 2.1, this is encapsured in Definition 1.

Definition 1 *A closed subset of $(S \times \mathbb{R}_{++}^L) \cap \mathcal{M}$ is called an anticipation set. Its elements are called anticipations, expectations or forecasts. We denote by \mathcal{A} the set of all anticipation sets. A collection $(P_i) \in \mathcal{A}^m$ is called an anticipation structure if:*

(a) $\cap_{i=1}^m P_i \neq \emptyset$.

We denote by \mathcal{AS} the set of anticipation structures. A structure, $(P'_i) \in \mathcal{AS}$, is said to refine, or to be a refinement of $(P_i) \in \mathcal{AS}$, and we denote it by $(P'_i) \leq (P_i)$, if:

(b) $P'_i \subset P_i, \forall i \in I$.

A refinement, $(P'_i) \in \mathcal{AS}$, of $(P_i) \in \mathcal{AS}$, is said to be self-attainable if:

(c) $\cap_{i=1}^m P'_i = \cap_{i=1}^m P_i$.

A belief is a probability, π , on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, whose support is an anticipation set, i.e., $P(\pi) \in \mathcal{A}$, using the notations of sub-Section 2.1. A structure of beliefs is a collection of beliefs, (π_i) , whose supports define an anticipation structure (i.e., $(P(\pi_i)) \in \mathcal{AS}$).

We denote by \mathcal{B} and \mathcal{SB} , respectively, the sets of beliefs and structures of beliefs. A structure of beliefs, $(\pi'_i) \in \mathcal{SB}$, is said to refine $(\pi_i) \in \mathcal{SB}$, which we denote $(\pi'_i) \leq (\pi_i)$, if the anticipation structure, $(P(\pi'_i))$, refines $(P(\pi_i)) \in \mathcal{AS}$ (i.e., $(P(\pi'_i)) \leq (P(\pi_i))$). A refinement, $(\pi'_i) \in \mathcal{SB}$, of $(\pi_i) \in \mathcal{SB}$ is said to be self-attainable if $\cap_{i=1}^m P(\pi'_i) = \cap_{i=1}^m P(\pi_i)$.

Remark 1 Along the above Definition, an anticipation set is a closed set of spot prices (at $t = 1$), whose values are never zero. A belief is a probability distribution on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, which cannot put a positive weight on arbitrarily low prices. Agents' anticipations or beliefs form a structure when they have some forecasts in common. The set of common forecasts is left unchanged at a self-attainable refinement.

2.4 Consumers' behavior and the notion of equilibrium

Agents implement their decisions at $t = 0$, when they reach their final anticipations, and elect related beliefs, $(\pi_i) \in \mathcal{SB}$, from observing market prices, $\omega_0 := (p_0, q) \in \mathcal{M}_0$, along a rational behavior described in Section 3, below. Hereafter, prices and beliefs at $t = 0$ are set as given, and markets, consistently, are assumed to have eliminated useless deals, so $\{(z_i) \in \mathbb{R}^{Jm} : \sum_{i=1}^m z_i = 0, V(\omega_i) \cdot z_i = 0, \forall (i, \omega_i) \in I \times P(\pi_i)\} = \{0\}$. Agent i 's consumption set is that of continuous mappings from $\{0\} \cup P(\pi_i)$ to \mathbb{R}_+^L ,

$$X(\pi_i) := \mathcal{C}(\{0\} \cup P(\pi_i), \mathbb{R}_+^L), \text{ for each } i \in I.$$

Thus, her consumptions, $x \in X(\pi_i)$, are mappings, relating $s = 0$ to a consumption decision, $x_0 := x_{\omega_0} \in \mathbb{R}_+^L$, at $t = 0$, and, continuously on $P(\pi_i)$, every expectation, $\omega := (s, p) \in P(\pi_i)$, to a consumption decision, $x_\omega \in \mathbb{R}_+^L$, at $t = 1$, which is conditional on the joint observation of state s , and price p , on the spot market.

Each agent $i \in I$ elects and implements a consumption and investment decision, or strategy, $(x, z) \in X(\pi_i) \times \mathbb{R}^J$, that she can afford on markets, given her endowment, $e_i \in \mathbb{R}_+^{LS'}$, and her expectation set, $P(\pi_i)$. This defines her budget set as follows:

$$B_i(\omega_0, \pi_i) := \{(x, z) \in X(\pi_i) \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leq -q \cdot z; \quad p_s \cdot (x_\omega - e_{i\omega}) \leq V(\omega) \cdot z, \forall \omega := (s, p_s) \in P(\pi_i)\}$$

An allocation, $(x_i) \in X[(\pi_i)] := \Pi_{i=1}^m X(\pi_i)$, is a collection of consumptions across consumers. We define the following set of attainable allocations:

$$A((\omega_s), (\pi_i)) := \{(x_i) \in X[(\pi_i)] : \sum_{i=1}^m (x_{i0} - e_{i0}) = 0, \sum_{i=1}^m (x_{i\omega_s} - e_{i\omega_s}) = 0, \forall s \in \underline{\mathbf{S}}, \text{ s.t. } \omega_s \in \cap_{i=1}^m P(\pi_i)\},$$

for every price collection, $(\omega_s) := (\omega_s)_{s \in \underline{\mathbf{S}}} \in \Pi_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$. Each agent $i \in I$ has preferences

represented by the V.N.M. utility function: $x \in X(\pi_i) \mapsto u_i^{\pi_i}(x) := \int_{\omega \in P(\pi_i)} u_i(x_0, x_\omega) d\pi_i(\omega).$

The generic i^{th} agent elects a strategy, which maximises her utility function in the budget set, i.e., a strategy of the set $B_i^*(\omega_0, \pi_i) := \arg \max_{(x,z) \in B_i(\omega_0, \pi_i)} u_i^{\pi_i}(x)$. The above economy is denoted by \mathcal{E} . Its equilibrium concept is defined as follows:

Definition 2 *A collection of prices, $(\omega_s) \in \Pi_{s \in \underline{S}} \mathcal{M}_s$, beliefs, $(\pi_i) \in \mathcal{SB}$, and strategies, $(x_i, z_i) \in B_i(\omega_0, \pi_i)$, defined for each $i \in I$, is a correct foresight equilibrium (C.F.E.), or sequential equilibrium (respectively, a temporary equilibrium) of the economy \mathcal{E} , if the following Conditions (a)-(b)-(c)-(d) (resp., Conditions (b)-(c)-(d)) hold:*

- (a) $\forall s \in \underline{S}, \omega_s \in \cap_{i=1}^m P(\pi_i)$;
- (b) $\forall i \in I, (x_i, z_i) \in B_i^*(\omega_0, \pi_i) := \arg \max_{(x,z) \in B_i(\omega_0, \pi_i)} u_i^{\pi_i}(x)$;
- (c) $(x_i) \in A((\omega_s), (\pi_i))$;
- (d) $\sum_{i=1}^m z_i = 0$.

Under above conditions, $(\pi_i) \in \mathcal{SB}$, or $(\omega_s) \in \Pi_{s \in \underline{S}} \mathcal{M}_s$, are said to support equilibrium.

We now present the arbitrage issue, which is closely related to that of existence.

3 No-arbitrage prices and the information they reveal

We define and characterize no-arbitrage prices and their revealing properties.

3.1 The model's no-arbitrage prices

Recalling the notations of sub-Section 2.1, we first define no-arbitrage prices.

Definition 3 *Let an anticipation set, $P \in \mathcal{A}$, a belief, $\pi \in \pi(P)$ (as denoted in 2.1), and price, $q \in \mathbb{R}^J$, be given. Price q is said to be a no-arbitrage price of P (or π), or P (or π) is said to be q -arbitrage-free, if the following condition holds:*

- (a) $\nexists z \in \mathbb{R}^J : -q \cdot z \geq 0$ and $V(\omega) \cdot z \geq 0, \forall \omega \in P(\pi) = P$, with one strict inequality;

We denote by $Q(P)$ (or $Q(\pi)$) the set of no-arbitrage prices of P (or π).

Let a structure, $(P_i) \in \mathcal{AS}$, and, for each $i \in I$, the above price set, $Q(P_i)$, be given.

We refer to $Q_c[(P_i)] := \cap_{i=1}^m Q(P_i)$ as the set of common no-arbitrage of $(P_i) \in \mathcal{AS}$.

The structure, $(P_i) \in \mathcal{AS}$, is said to be arbitrage-free (respectively, q -arbitrage-free) if $Q_c[(P_i)]$ is non-empty (resp., if $q \in Q_c[(P_i)]$).

We say that q is a no-arbitrage price (resp., a self-attainable no-arbitrage price) of $(P_i) \in \mathcal{AS}$ if there exists a refinement (resp., self-attainable refinement), (P_i^*) , of (P_i) , such that $q \in Q_c[(P_i^*)]$. We denote by $Q[(P_i)]$ the set of no-arbitrage prices of (P_i) .

The above definitions and notations extend to any consistent structure of beliefs, $(\pi_i) \in \Pi_{i=1}^m \pi(P_i)$, as denoted in sub-Section 2.1. We then refer to $Q_c[(\pi_i)] := Q_c[(P_i)]$ and $Q[(\pi_i)] := Q[(P_i)]$ as, respectively, the sets of common no-arbitrage prices, and no-arbitrage prices, of the structure $(\pi_i) \in \mathcal{SB}$.

Remark 2 A symmetric refinement of any structure $(P_i) \in \mathcal{AS}$, that is, $(P'_i) \leq (P_i)$, such that $P'_i = P'_1$, for every $i \in I$, is always arbitrage-free along Definition 3. Hence, any structure, $(P_i) \in \mathcal{AS}$, admits a self-attainable no-arbitrage price. Indeed, the symmetric refinement, $(P_i^*) \leq (P_i)$, such that $P_1^* = \cap_{i=1}^m P_i$ is arbitrage-free.

Claim 1 states a simple but useful property of arbitrage-free structures.

Claim 1 *An arbitrage-free structure, $(P_i) \in \mathcal{AS}$, satisfies the following Assertion:*

(i) $\nexists (z_i) \in \mathbb{R}^{J^m} : \sum_{i=1}^m z_i = 0$ and $V(\omega) \cdot z \geq 0, \forall \omega \in \cup_{i=1}^m P_i$, with one strict inequality.

Proof Let $(P_i) \in \mathcal{AS}$ be an arbitrage-free anticipation structure and $q \in Q_c[(P_i)] \neq \emptyset$ be given. Assume, by contraposition, that there exists $(z_i) \in (\mathbb{R}^J)^m$, such that $\sum_{i=1}^m z_i = 0$ and $V(\omega) \cdot z \geq 0$, for every $\forall \omega \in \cup_{i=1}^m P_i$, with one strict inequality, say for $\omega \in P_1$. If $q \cdot z_1 \leq 0$, then, $q \notin Q(P_1)$, which contradicts the fact that $q \in Q_c[(P_i)]$. Hence, $q \cdot z_1 > 0$, which implies, from the relation $\sum_{i=1}^m z_i = 0$, that $q \cdot z_i < 0$, for some $i \in I$. Then, from above, $q \notin Q(P_i)$, which also contradicts the fact that $q \in Q_c[(P_i)]$. \square

3.2 Individual anticipations revealed by prices

Claim 2 tackles the notion of information conveyed by prices to individual agents.

Claim 2 *Let $(P_i) \in \mathcal{AS}$, and $q \in \mathbb{R}^J$, be given. Then, for each $i \in I$, there exists a set, $\bar{P}_i(q) \in \emptyset \cup \mathcal{A}$, said to be revealed by price q to agent i , such that:*

- (i) *if $\bar{P}_i(q) \neq \emptyset$, then, $\bar{P}_i(q) \subset P_i$ and $\bar{P}_i(q)$ is q -arbitrage-free;*
- (ii) *every q -arbitrage-free anticipation set included in P_i is a subset of $\bar{P}_i(q)$.*

Proof Let $i \in I$, $q \in \mathbb{R}^J$ and an anticipation structure, $(P_i) \in \mathcal{AS}$, be given. Let $\mathcal{R}_{(P_i, q)}$ be the set of q -arbitrage-free anticipation sets included in P_i . If $\mathcal{R}_{(P_i, q)} = \emptyset$, the set $\bar{P}_i(q) = \emptyset$ meets the conditions of Claim 2. We henceforth assume that $\mathcal{R}_{(P_i, q)} \neq \emptyset$ and let $\{P_i^k\}_{k \in K}$ be a chain of $\mathcal{R}_{(P_i, q)}$, that is, a totally ordered (for the inclusion relation) set of elements of $\mathcal{R}_{(P_i, q)}$. We recall P_i is closed. By construction, $P_i^* := \overline{\cup_{k \in K} P_i^k}$ is an anticipation set, such that $P_i^k \subset P_i^* \subset P_i$, for every $k \in K$.

Assume, by contraposition, that $P_i^* \notin \mathcal{R}_{(P_i, q)}$, i.e., there exists $z \in \mathbb{R}^J$ and $\bar{\omega} \in P_i^*$, such that $-q \cdot z \geq 0$, $V(\omega) \cdot z \geq 0$ for every $\omega \in P_i^*$, and $(V(\bar{\omega}) \cdot z - q \cdot z) > 0$. Since, $P_i^k \subset P_i^*$ is q -arbitrage-free for every $k \in K$, the above relations imply, from Definition 3, that $-q \cdot z = 0$ and $V(\omega^k) \cdot z = 0$, for every $(k, \omega^k) \in K \times P_i^k$. Since V is continuous, the relation $V(\bar{\omega}) \cdot z > 0$, which holds for $\bar{\omega} \in P_i^* := \overline{\cup_{k \in K} P_i^k}$, implies the existence of $k \in K$ and $\omega_i^k \in P_i^k$, such that $V(\omega_i^k) \cdot z > 0$, which contradicts the above equalities. Hence, $P_i^* \in \mathcal{R}_{(P_i, q)}$. From above, P_i^* is an upper bound of the chain $\{P_i^k\}_{k \in K}$ in $\mathcal{R}_{(P_i, q)}$. Hence, from Zorn's Lemma (see Aliprantis-Border, 1999, p. 14), $\mathcal{R}_{(P_i, q)}$ has a maximal element, that is, contains a set, $\bar{P}_i(q) \neq \emptyset$, satisfying Claim 2-(i)-(ii). \square

3.3 Anticipation structures revealed by prices

The following Claim characterizes the no-arbitrage prices of Definition 3.

Claim 3 *Let a price, $q \in \mathbb{R}^J$, an anticipation structure, $(P_i) \in \mathcal{AS}$, and the related set collection, $(\bar{P}_i(q))$, of Claim 2, be given. The following statements are equivalent:*

- (i) *q is a no-arbitrage price of (P_i) ;*
- (ii) *$(\bar{P}_i(q))$ is the coarsest q -arbitrage-free refinement of (P_i) ;*
- (iii) *$(\bar{P}_i(q))$ is a refinement of (P_i) ;*
- (iv) *$(\bar{P}_i(q))$ is an anticipation structure.*

If $q \in Q[(P_i)]$ is self-attainable, the above refinement, $(\bar{P}_i(q)) \leq (P_i)$, is self-attainable.

Proof (i) \Rightarrow (ii) Let $q \in Q[(P_i)]$ be given. From Definition 3, we set as given a q -arbitrage-free refinement, (P_i^*) , of (P_i) . Then, for each $i \in I$, the set, $\mathcal{R}_{(P_i, q)}$, of q -arbitrage-free anticipation sets included in P_i is non-empty (it contains P_i^*), and, from Claim 2, admits $\bar{P}_i(q) \neq \emptyset$ for maximal element, that is, $P_i^* \subset \bar{P}_i(q) \subset P_i$ and $q \in Q[(\bar{P}_i(q))]$. These relations imply that $(P_i^*) \leq (\bar{P}_i(q)) \leq (P_i)$ and $q \in Q_c[(\bar{P}_i(q))]$, i.e., $(\bar{P}_i(q))$ is the coarsest q -arbitrage-free refinement of (P_i) , since (P_i^*) was set arbitrary.

(ii) \Rightarrow (iii) \Rightarrow (iv) The relations are immediate from Definition 1.

(iv) \Rightarrow (i) If $(\bar{P}_i(q)) \in \mathcal{AS}$, then, from Claim 2, $(\bar{P}_i(q))$ refines (P_i) and is q -arbitrage-free, that is, $q \in Q_c[(\bar{P}_i(q))] \subset Q[(P_i)]$.

The end of Claim 3, left to readers, is immediate from Definition 1 and above. \square

Claim 3 permits to define concepts of revealed and price-revealable refinements.

Definition 4 *Let an anticipation structure, $(P_i) \in \mathcal{AS}$, and a no-arbitrage price, $q \in Q[(P_i)]$, be given. The refinement, $(\bar{P}_i(q)) \leq (P_i)$, of Claim 3 is said to be revealed by price q . A refinement, (P'_i) , of (P_i) is said to be price-revealable if it can be revealed by some price, i.e., there exists $q' \in Q[(P_i)]$ such that $(P'_i) = (\bar{P}_i(q'))$ along Claim 3. Consistently, we say that an arbitrage-free anticipation structure is price revealable and revealed by any of its common no-arbitrage prices.*

We now examine how agents, endowed with no price model a la Radner, may still update their anticipations from observing market prices. The following sub-Section generalizes a result of Cornet-de Boisdeffre (2009) to the above economy, \mathcal{E} .

3.4 Sequential refinement through prices

Throughout, we let an anticipation structure, $(P_i) \in \mathcal{AS}$, a generic agent, $i \in I$, and an asset price, $q \in \mathbb{R}^J$, be given. We study how this i^{th} agent, endowed with the initial set of anticipations, P_i , may update her forecasts from observing price q .

We thus define, by induction, two sequences, $\{A_i^n\}_{n \in \mathbb{N}}$ and $\{P_i^n\}_{n \in \mathbb{N}}$ as follows:

- for $n = 0$, we let $A_i^0 = \emptyset$ and $P_i^0 := P_i$;
- for $n \in \mathbb{N}$ arbitrary, with A_i^n and P_i^n defined at step n , we let $A_i^{n+1} := P_i^{n+1} := \emptyset$, if $P_i^n = \emptyset$, and, otherwise,

$$A_i^{n+1} := \{\bar{\omega} \in P_i^n : \exists z \in \mathbb{R}^J, -q \cdot z \geq 0, V(\bar{\omega}) \cdot z > 0 \text{ and } V(\omega) \cdot z \geq 0, \forall \omega \in P_i^n\};$$

$$P_i^{n+1} := P_i^n \setminus A_i^{n+1}, \text{ i.e., the agent rules out anticipations, granting an arbitrage.}$$

Claim 4 *Let an anticipation structure, $(P_i) \in \mathcal{AS}$, an agent, $i \in I$, a price, $q \in \mathbb{R}^J$, and the information set, $\bar{P}_i(q)$, it reveals along Claim 2, be given. The above set sequences, $\{A_i^n\}_{n \in \mathbb{N}}$ and $\{P_i^n\}_{n \in \mathbb{N}}$, satisfy the following assertions:*

- (i) $\exists N \in \mathbb{N} : \forall n > N, A_i^n = \emptyset \text{ and } P_i^n = P_i^N$;
- (ii) $P_i^N = \lim_{n \rightarrow \infty} P_i^n = \bar{P}_i(q)$.

Proof Let $(P_i) \in \mathcal{AS}$, $i \in I$, $q \in \mathbb{R}^J$, $\bar{P}_i(q)$, $\{A_i^n\}_{n \in \mathbb{N}}$ and $\{P_i^n\}_{n \in \mathbb{N}}$ be set or defined as in Claim 4, and let $P_i^* := \bigcap_{n \in \mathbb{N}} P_i^n = \lim_{n \rightarrow \infty} \searrow P_i^n$.

With a non-restrictive convention that the empty set be included in any other set, we show, first, that the inclusion $\bar{P}_i(q) \subset P_i^n$ holds for every $n \in \mathbb{N}$. It holds, from

Claim 2, for $n = 0$ (since $\bar{P}_i(q) \subset P_i^0 = P_i$). Assume, now, by contraposition, that, for some $n \in \mathbb{N}$, $\bar{P}_i(q) \subset P_i^n$ and $\bar{P}_i(q) \not\subset P_i^{n+1}$. Then, there exist $\bar{\omega} \in \bar{P}_i(q) \cap A_i^{n+1}$ and $z \in \mathbb{R}^J$, such that $-q \cdot z \geq 0$, $V(\bar{\omega}) \cdot z > 0$ and $V(\omega) \cdot z \geq 0$, for every $\omega \in \bar{P}_i(q) \subset P_i^n$, which contradicts Claim 2, along which $\bar{P}_i(q)$ is q -arbitrage-free, if non-empty. Hence, the relation $\bar{P}_i(q) \subset P_i^n$, holds for every $n \in \mathbb{N}$.

Assume, first, that $P_i^* := \bigcap_{n \in \mathbb{N}} P_i^n = \emptyset$. Since the sequence $\{P_i^n\}_{n \in \mathbb{N}}$ is non-increasing and made of compact or empty sets (this stems, by induction, from the fact that P_i^0 is compact and A^n is open or empty), there exists $N \in \mathbb{N}$, such that $P_i^n = A_i^n = \emptyset$, for every $n \geq N$, and, from above, assertions (i)-(ii) of Claim 4 hold (with $\bar{P}_i(q) = \emptyset$).

Assume, next, that $P_i^* \neq \emptyset$. Then, P_i^* , a non-empty intersection of compact sets, is compact, and, from above, $\bar{P}_i(q) \subset P_i^*$.

For every $n \in \mathbb{N}$, let $Z_i^{on} := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in P_i^n\}$. Since $\{P_i^n\}_{n \in \mathbb{N}}$ non-increasing, the sequence of vector spaces, $\{Z_i^{on}\}$, is non-decreasing in \mathbb{R}^J , hence, stationary. We let $N \in \mathbb{N}$ be such that $Z_i^{on} = Z_i^{oN}$, for every $n \geq N$. Assume, by contraposition, that assertion (i) of Claim 4 fails, that is:

$$\forall n \in \mathbb{N}, \exists (\omega_n, z_n) \in P_i^n \times \mathbb{R}^J : -q \cdot z_n \geq 0, V(\omega_n) \cdot z_n > 0 \text{ and } V(\omega) \cdot z_n \geq 0, \forall \omega \in P_i^n.$$

From the definition of the sets P_i^n and $P_i^{n+1} \neq \emptyset$, the above potfolios satisfy, for each $n \geq N$, jointly $z_n \notin Z_i^{on}$ and $z_n \in Z_i^{o(n+1)}$, which contradicts the fact $Z_i^{on+1} = Z_i^{on}$. This contradiction proves that assertion (i) of Claim 4 holds, and we let $N \in \mathbb{N}$ be such that $A_i^{N+1} = \emptyset$. Then, by construction, $P_i^N = P_i^*$, and $P_i^* \subset P_i$ is q -arbitrage-free (for $A_i^{N+1} = \emptyset$), which yields, from Claim 2, $P_i^* \subset \bar{P}_i(q)$, and, from above, $P_i^* = \bar{P}_i(q)$. \square

The above inference process is a rational behavior, that we refer to as the no-arbitrage principle, whereby agents, having no clue of how market prices are determined, update their beliefs from observing them and ruling out arbitrage. They

reach their final update after finitely many inference steps. As long as markets have not reached a no-arbitrage price, traders cannot agree on prices and a sequential equilibrium may not exist. Claims 3 and 4 show that agents have common updated forecasts - a necessary condition of the sequential equilibrium - if, and only if, the observed asset price is a no-arbitrage price. We have seen such prices always exist. We will see below that equilibrium prices are always no-arbitrage prices, hence, agents may infer their own anticipations from the observed equilibrium price. We then speak of price-revealed equilibria. We now introduce and discuss the notion of minimum price uncertainty of a private belief economy, and state our Theorem.

4 An uncertainty principle and the existence of equilibrium

4.1 The existence Theorem

In a private belief economy, there exists a set of minimum uncertainty about future prices, any element of which can obtain as an equilibrium price for some structure of beliefs today.

Definition 5 *Let Ω be the set of sequential equilibria (CFE) of the economy, \mathcal{E} . The minimum uncertainty set, Δ , is the subset of prices at $t = 1$, which support a CFE:*

$$\Delta = \{\omega^* = (s^*, p^*) \in \mathcal{M} : s^* \in \underline{\mathbf{S}}, \exists((\omega_s), (\pi_i), [(x_i, z_i)]) \in \Omega, \omega^* = \omega_{s^*}\}.$$

In Section 5, we will prove that Δ is non-empty and that a continuum of equilibria exists in standard conditions. Thus, Δ is typically uncountable. Rational agents' forecasts of spot prices cannot degenerate to singletons, unless all agents' forecasts are the same and common knowledge. In the economy, \mathcal{E} , agents' information and beliefs are private. So, each agent, $i \in I$, should anticipate a typically uncountable set, $P_i \in \mathcal{A}$, of possible prices on each spot market tomorrow, whose eventual

realization depends on all agents' private beliefs today. We retain the standard small consumer, price-taker, hypothesis, which states that no single agent's belief, or strategy, may have a significative impact on equilibrium prices. The economy, \mathcal{E} , is said to be standard if, moreover, it meets the following Conditions:

- **Assumption A1:** $\forall i \in I, e_i >> 0$;
- **Assumption A2:** $\forall i \in I, u_i$ is class C^1 , strictly concave, strictly increasing.

The following Theorem states that a standard economy, \mathcal{E} , admits a non-empty set, Δ , of minimum uncertainty, and that a C.F.E. exists, whatever their beliefs, as long as agents' anticipations embed the set Δ .

Theorem 1 *Let a standard economy, \mathcal{E} , and an anticipation structure, $(P_i) \in \mathcal{AS}$, be given. Let Δ be the minimum uncertainty set. The following Assertions hold:*

- (i) $\exists \varepsilon > 0 : \forall (s, p) \in \Delta, \forall l \in \mathcal{L}, p^l \geq \varepsilon$;
- (ii) $\Delta \neq \emptyset$.

Consistently with Assertions (i)-(ii) above, if the relation $\Delta \subset \cap_{i=1}^m P_i$ holds, that is, if every agent's anticipations embed Δ , then, referring to the notations of sub-Section 2.1, the following Assertions hold:

- (iii) *if the structure (P_i) is arbitrage-free, any beliefs $(\pi_i) \in \Pi_{i=1}^m \pi(P_i)$ support a CFE;*
- (iv) *if a self-attainable refinement, $(P_i^*) \leq (P_i)$, is arbitrage-free (and such refinements exist), any refinement of beliefs, $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$, as denoted in 2.1, supports a CFE;*
- (v) *if $(P_i^*) \in \mathcal{AS}$ is a self-attainable price-revealable refinement of (P_i) (which exists), then, every refinement $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$ supports a price-revealed CFE.*

Remark 3 Assertion (iv) of Theorem 1 is a direct Corollary of assertion (iii), from replacing the structure (P_i) by the arbitrage-free structure (P_i^*) . We let the reader check, as standard from Assumption A2, that if $(\pi_i^*) \in \mathcal{SB}$ and $\omega_0 := (p_0, q) \in \mathcal{M}_0$

support a C.F.E., then, $q \in Q_c[(\pi_i^*)]$. Consequently, if that supporting structure, $(\pi_i^*) \in \mathcal{SB}$, is price-revealable, then, it is revealed by the equilibrium price along Definition 4, i.e., the CFE is price-revealed. Given this, Assertion (v) of Theorem 1 is a Corollary of Assertion (iv) and only Assertions (i)-(ii)-(iii) need be proved.

Before discussing the Theorem' Condition, $\Delta \subset \cap_{i=1}^m P_i$, we prove Assertion (i).

Proof of Assertion (i) Let Ω and Δ be the sets of Definition 5, and let $\omega^* := (s^*, p^*) \in \Delta$, for some $s^* \in \underline{\mathbf{S}}$, be given. Then, there exist prices, $(\omega_s) \in \Pi_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$, beliefs, $(\pi_i) \in \mathcal{SB}$, and strategies, $[(x_i, z_i)] \in \Pi_{i=1}^m B_i(\omega_0, \pi_i)$, such that $\mathcal{C} := ((\omega_s), (\pi_i), [(x_i, z_i)]) \in \Omega$ and $\omega^* = \omega_{s^*}$. Let $e := \max_{(s,l) \in S' \times \mathcal{L}} \sum_{i=1}^m e_{is}^l > 0$ and $\beta := \sup \frac{\partial u_i}{\partial y^{l'}}(x, y) / \frac{\partial u_i}{\partial y^l}(x, y)$, for $(i, (x, y), (l, l')) \in I \times [0, e]^{2L} \times \mathcal{L}^2$, be given. Then, for each $s \in \underline{\mathbf{S}}'$, the relations $(x_{is}) \geq 0$ and $\sum_{i=1}^m (x_{is} - e_{is}) = 0$ hold, from Condition (c) of Definition 2, and imply $x_{is} \in [0, e]^L$, for each $i \in I$. Moreover, the relations $\beta \in \mathbb{R}_{++}$ and $p \in S \times \mathbb{R}_{++}^L$ are standard from Assumption A2 and Condition (b) of Definition 2 (applied to \mathcal{C}). Let $(l, l') \in \mathcal{L}^2$ be given. We show that $\frac{p^l}{p^{l'}} \leq \beta$. Otherwise, it is standard, from Assumptions A1-A2 and Conditions (b)-(c) of Definition 2, that there exist $i \in I$, such that $x_i^l > 0$, and $y \in X(\pi_i)$, such that $\omega \mapsto y_\omega$ is identical to $\omega \mapsto x_{i\omega}$, but on a small neighborhood of ω^* , where $y_{\omega^*}^{l'} = x_{i\omega^*}^{l'} + \frac{\delta}{p^{l'}} \neq x_{i\omega^*}^{l'}$ and $y_{\omega^*}^l = x_{i\omega^*}^l - \frac{\delta}{p^l} \neq x_{i\omega^*}^l$ (for $\delta \in \mathbb{R}_{++}$ small enough), which satisfy $(y, z_i) \in B_i(\omega_0, \pi_i)$ and $u_i^{\pi_i}(y) > u_i^{\pi_i}(x_i)$, thus contradicting the fact that \mathcal{C} meets Definition 2-(b). Then, we let the reader check from the joint relations $p > 0$, $\|p\| = 1$ and $\frac{p^l}{p^{l'}} \leq \beta$ (which hold for each $(l, l') \in \mathcal{L}^2$), that $p^l \geq \varepsilon = \frac{1}{\beta \sqrt{L}}$, for each $l \in \mathcal{L}$. \square

We now discuss the Theorem's Condition, encapsulating an uncertainty principle.

4.2 The Theorem's Condition

The Theorem's Condition, $\Delta \subset \cap_{i=1}^m P_i$, is consistent with the model's assumption of price-takers agents seeing other consumers' beliefs as arbitrary. Then, Δ , the

set of all possible equilibrium prices, for some structure of beliefs today (which no agent knows), may be seen as a set incompressible uncertainty. From Theorem 1, the Condition, $\Delta \subset \cap_{i=1}^m P_i$, is sufficient to insure the existence of a CFE. It may also be a necessary one, if beliefs are effectively unpredictable and erratic enough to let any price in Δ become a possible outcome. This situation may arise in times of enhanced uncertainty, volatility or erratic change in beliefs, which lets no room for coordination or agreement in anticipating a somewhat unpredictable future.

If cautious agents should embed the minimum uncertainty set into their anticipations, the question arises why and how this should happen. An empirical answer may be that *relative* prices are often observable, on long time series, for virtually all types of uncertainties, beliefs or states of nature, and would vary between observable boundaries. From the relative prices so observed, the set, Δ , of tomorrow's realizable equilibrium prices, or a bigger set, might be inferred by the collectivity. In addition, agents may have an idiosyncratic uncertainty, given their personal information and beliefs, so their anticipation sets need not reduce to Δ or be symmetric.

We showed in Section 3 that agents, starting from any anticipation structure, $(P_i) \in \mathcal{AS}$, could always reach an arbitrage-free refinement, $(P_i^*) \leq (P_i)$, from observing market no-arbitrage prices. Without a price model, agents cannot infer more. From Theorem 1, once they have reached the refinement (P_i^*) , all possible equilibrium prices at $t = 0$ are related to beliefs, $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$, which have the same supports, namely, $(P(\pi_i^*)) = (P_i^*)$. So, spot prices at $t = 0$ are no longer informative. The refined anticipation sets of Definition 4, (P_i^*) , are agents' final information. Under the Theorem's Condition, that information is always consistent with a price revealed equilibrium, no matter agents' probability distributions on final sets, (P_i^*) .

In all cases, agents' final beliefs, $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$, remain private. This privacy,

agents' fixed expectations sets, (P_i^*) , and the Theorem's Condition restore existence. There can be no fall in rank problem a la Hart (1975). The generic i^{th} agent's budget set and strategy are defined *ex ante*, with reference to ex ante conditions, and to a fixed set of anticipations, P_i^* . So, only her *ex ante span* of payoffs matters, namely, $\langle V, P_i^* \rangle := \{\omega \in P_i^* \mapsto V(\omega) \cdot z : z \in \mathbb{R}^J\}$. That span is fixed independently of any equilibrium price, $p \in \Delta \subset P_i^*$, whose location in the set Δ cannot be predicted at $t = 0$ and will only be observed at $t = 1$. This setting is quite different from Hart's.

5 The existence proof

Throughout, we set as given arbitrarily, in a standard economy, \mathcal{E} , an arbitrage-free anticipation structure, (P_i) and related beliefs, $(\pi_i) \in \Pi_{i=1}^m \pi(P_i)$ (using sub-Section 2.1' notations). They are, henceforth, fixed and always referred to. Along Remark 3, we need only prove assertions (ii) and (iii) of Theorem 1. The proof's principle is to construct a sequence of auxiliary economies, with finite anticipation sets, refining and tending to the initial sets, (P_i) . Each finite economy admits an equilibrium, which we set as given along Theorem 1 of [6]. Then, from the sequence of finite dimensional equilibria, we derive an equilibrium of the initial economy, \mathcal{E} .

5.1 Auxiliary sets

We divide P_i , for each $i \in I$, in ever finer partitions, and let for each $n \in \mathbb{N}$:

$$K_n := \{k_n := (k_n^1, \dots, k_n^L) \in (\mathbb{N} \cap [0, 2^n - 1])^L\};$$

$$P_{(i,s,k_n)} := P_i \cap (\{s\} \times \Pi_{l \in \{1, \dots, L\}}]\frac{k_n^l}{2^n}, \frac{k_n^l + 1}{2^n}]), \text{ for every } (s, k_n := (k_n^1, \dots, k_n^L)) \in S_i \times K_n.$$

For each $(i, s, n, k_n) \in I \times S_i \times \mathbb{N} \times K_n$, such that $P_{(i,s,k_n)} \neq \emptyset$, we select $g_{(i,s,k_n)}^n \in P_{(i,s,k_n)}$ uniquely, and define a set, $G_i^n := \{g_{(i,s',k'_n)}^n : s' \in S_i, k'_n \in K_n, P_{(i,s',k'_n)} \neq \emptyset\}$, as follows:

- for $n = 0$, we select one $g_{(i,s,0)}^0 \in P_{(i,s,0)}$, for all $s \in S_i$, and let $G_i^0 := \{g_{(i,s,0)}^0 : s \in S_i\}$;
- for $n \in \mathbb{N}^*$ arbitrary, given $G_i^{n-1} := \{g_{(i,s,k_{n-1})}^{n-1} \in P_{(i,s,k_{n-1})} : s \in S_i, k_{n-1} \in K_{n-1}\}$, we let, for every $(s, k_n) \in S_i \times K_n$, such that $P_{(i,s,k_n)} \neq \emptyset$,

$$g_{(i,s,k_n)}^n \begin{cases} \text{be equal to } g_{(i,s,k_{n-1})}^{n-1}, & \text{if there exists } (k_{n-1}, g_{(i,s,k_{n-1})}^{n-1}) \in K_{n-1} \times G_i^{n-1} \cap P_{(i,s,k_n)} \\ \text{be set fixed in } P_{(i,s,k_n)}, & \text{if } G_i^{n-1} \cap P_{(i,s,k_n)} = \emptyset \end{cases}^2$$

This yields a set, $G_i^n := \{g_{(i,s,k_n)}^n \in P_{(i,s,k_n)} : s \in S_i, k_n \in K_n\}$, and, by induction, a non-decreasing dense sequence, $\{G_i^n\}_{n \in \mathbb{N}}$, of subsets of P_i , with a good property:

Lemma 1 *There exists $N \in \mathbb{N}$, such that the following Assertion holds:*

(i) $\forall (P_i^*) \leq (P_i), (G_i^N \subset P_i^*), \forall i \in I \Rightarrow (Q_c[(P_i^*)] \neq \emptyset)$.

Proof see the Appendix. □

For every integer $n \geq N$ along Lemma 1, and any element $\eta \in]0, 1]$, hereafter set as given, we consider an auxiliary economy, \mathcal{E}_η^n , which admits an equilibrium, \mathcal{C}^n .

5.2 Auxiliary economies, \mathcal{E}_η^n

We let $\mathbb{N}_N := \mathbb{N} \setminus \{0, 1, \dots, N-1\}$, along Lemma 1, and set as given, for each $s \in \underline{\mathbf{S}}$, an arbitrary spot price, $\omega_s^{N-1} := (s, p_s^{N-1}) \in \mathcal{M}_s$. Then, we define, by induction on $n \in \mathbb{N}_N$, a sequence of prices, $\{(\omega_s^n)\} \in (\Pi_{s \in \underline{\mathbf{S}}} \mathcal{M}_s)^{\mathbb{N}_N}$, which are, for each $n \in \mathbb{N}_N$, the second period equilibrium prices of the economy \mathcal{E}_η^n , presented hereafter.

We now let $n \in \mathbb{N}_N$ be given and derive from the set, G_i^n , of sub-Section 5.1, and prices, $(\omega_s^{n-1}) \in \Pi_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$, assumed to be defined at the last induction step, an auxiliary economy, \mathcal{E}_η^n , referred to as the (n, η) -economy, which is of the type described in [6]. Namely, it is a pure exchange economy, with two period ($t \in \{0, 1\}$),

² Non restrictively (up to a shift in the upper boundary of $P_{(i,s,k_n)}$), we assume that each $g_{(i,s,k_n)}^n \in P_{(i,s,k_n)}$ is in the interior of $P_{(i,s,k_n)} \neq \emptyset$, to insure that $\pi_i(P_{(i,s,k_n)}) > 0$.

m agents, having incomplete information, and exchanging L goods and J nominal assets, under uncertainty (at $t = 0$) about which state of a finite state space, S^n , will prevail at $t = 1$. Referring to [6], and to the above notations and definitions in the economy \mathcal{E} , the generic (n, η) -economy's characteristics are as follows:

- The information structure is the collection, (S_i^n) , of sets $S_i^n := \underline{\mathbf{S}} \cup \tilde{S}_i^n$ (and we let $S_i'^n := \underline{\mathbf{S}}' \cup \tilde{S}_i^n$), such that $\tilde{S}_i^n := \{i\} \times G_i^n$ is defined for each $i \in I$. The pooled information set (of the states which may prevail at $t = 1$) is, hence, $\underline{\mathbf{S}} = \cap_{i \in I} S_i^n$. For each $i \in I$, the set $\tilde{S}_i^n := \{i\} \times G_i^n$ consists of purely formal states, none of which will prevail. The state space of the (n, η) -economy is $S^n = \cup_{i \in I} S_i^n$.
- The $S^n \times J$ payoff matrix, $V^n := (V^n(s^n))$, is defined, with reference to the payoff mapping, V , of the economy \mathcal{E} , by $V^n(s) := V((s, p_s^{n-1}))$, for each $s \in \underline{\mathbf{S}}$, and $V^n(s^n) := V(\omega)$, for each $s^n := (i, \omega) \in S^n$. The payoff matrix V^n is purely nominal.
- In each formal state, $s^n := (i, (s, p_s)) \in \tilde{S}_i^n$, the generic agent $i \in I$ is certain that price $p_s \in \mathbb{R}_{++}^L$, and only that price, can prevail on the s^n -spot market.
- In each realizable state, $s \in \underline{\mathbf{S}}$, the generic agent $i \in I$ has perfect foresight, i.e., anticipates with certainty the true price, say $p_s^n \in \mathbb{R}_{++}^L$ (or $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$).
- The generic i^{th} agent's endowment, $e_i^n := (e_{is^n}^n) \in \mathbb{R}_{++}^{LS_i'^n}$, is defined (with reference to e_i in \mathcal{E}) by $e_{is}^n := e_{is}$, for each $s \in \underline{\mathbf{S}}'$, and $e_{is^n}^n := e_{is}$, for each $s^n := (i, (s, p_s)) \in \tilde{S}_i^n$.
- For every collection of the true market prices, $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$, at $t = 0$, and $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$, for all $s \in \underline{\mathbf{S}}$, at $t = 1$, the generic i^{th} agent has the following consumption set, X_i^n , budget set, $B_i^n(S_i^n, (\omega_s^n))$, and utility function, u_i^n :

$$X_i^n := \mathbb{R}_+^{LS_i'^n};$$

$$B_i^n(S_i^n, (\omega_s^n)) := \left\{ (x, z) \in X_i^n \times \mathbb{R}^J : \begin{cases} p_0^n \cdot (x_0 - e_{i0}) \leq -q^n \cdot z \quad \text{and} \quad p_s^n \cdot (x_s - e_{is}) \leq V^n(s) \cdot z, \forall s \in \underline{\mathbf{S}} \\ p_s \cdot (x_{s^n} - e_{is^n}) \leq V^n(s^n) \cdot z, \forall s^n := (i, (s, p_s)) \in \tilde{S}_i^n \end{cases} \right\};$$

$$u_i^n : x \mapsto \sum_{s^n \in S_i^n} \pi_i^n(s^n) u_i(x_0, x_{s^n}),$$

where $\pi_i^n(s^n) := \pi_i(P_{(i,s,k_n)})$, for every $(s, k_n, s^n) \in S_i \times K_n \times (\{i\} \times [G_i^n \cap P_{(i,s,k_n)}])$, is the probability of the set $\underline{P_{(i,s,k_n)} \neq \emptyset}$, along the belief π_i , and $\pi_i^n(s) := \eta$, for each $s \in \underline{\mathbf{S}}$.

Theorem 1 of [6] and Lemma 1 above yield Lemma 2.

Lemma 2 *The generic (n, η) -economy admits an equilibrium, namely a collection of prices, $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$, at $t = 0$, and $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$, in each state $s \in \underline{\mathbf{S}}$, and strategies, $(x_i^n, z_i^n) \in B_i^n(S_i^n, (\omega_s^n))$, defined for each $i \in I$, such that:*

$$(i) \forall i \in I, (x_i^n, z_i^n) \in \arg \max_{(x,z) \in B_i^n(S_i^n, (\omega_s^n))} u_i^n(x);$$

$$(ii) \forall s \in \underline{\mathbf{S}}', \sum_{i=1}^m (x_{is}^n - e_{is}) = 0;$$

$$(iii) \sum_{i=1}^m z_i^n = 0.$$

Moreover, the equilibrium prices and allocations satisfy the following Assertions:

$$(iv) \forall (n, i, s) \in \mathbb{N}_N \times I \times \underline{\mathbf{S}}', x_{is}^n \in [0, e]^L, \text{ where } e := \max_{(s,l) \in S' \times \mathcal{L}} \sum_{i=1}^m e_{is}^l;$$

$$(v) \exists \varepsilon \in]0, 1] : p_s^{nl} \geq \varepsilon, \forall (n, s, l) \in \mathbb{N}_N \times \underline{\mathbf{S}} \times \mathcal{L}.$$

Proof see the Appendix. □

Along Lemma 2, we set as given an equilibrium of the (n, η) -economy, namely:

$$\mathcal{C}^n := (\omega_0^n := (p_0^n, q^n), (\omega_s^n), [(x_i^n, z_i^n)]) \in \mathcal{M}_0 \times \prod_{s \in \underline{\mathbf{S}}} \mathcal{M}_s \times \prod_{i=1}^m B_i^n(S_i^n, (\omega_s^n)),$$

which is always referred to. The equilibrium prices, $(\omega_s^n) \in \prod_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$, permit to pursue the induction and define the $(n+1, \eta)$ -economy in the same way as above, hence, the auxiliary economies and equilibria at all ranks. These meet the following Lemma.

Lemma 3 *For the above sequence, $\{\mathcal{C}^n\}$, of equilibria, it may be assumed to exist:*

$$(i) \omega_0^\eta := (p_0^\eta, q^\eta) = \lim_{n \rightarrow \infty} \omega_0^n \in \mathcal{M}_0 \text{ and } \omega_s^\eta := (s, p_s^\eta) = \lim_{n \rightarrow \infty} \omega_s^n \in \mathcal{M}_s, \text{ for each } s \in \underline{\mathbf{S}};$$

$$(ii) (x_{is}^\eta) := \lim_{n \rightarrow \infty} (x_{is}^n)_{i \in I} \in \mathbb{R}^{Lm}, \text{ such that } \sum_{i \in I} (x_{is}^\eta - e_{is}) = 0, \text{ for each } s \in \underline{\mathbf{S}}';$$

(iii) $(z_i^\eta) = \lim_{n \rightarrow \infty} (z_i^n)_{i \in I} \in \mathbb{R}^{J^m}$, such that $\sum_{i=1}^m z_i^\eta = 0$.

Moreover, we define, for each $i \in I$, the following sets and mappings:

- * $G_i^\infty := \cup_{n \in \mathbb{N}} G_i^n = \lim_{n \rightarrow \infty} \nearrow G_i^n \subset P_i$;
- * for each $n \in \mathbb{N}$, the mapping, $\omega \in P_i \mapsto \arg_i^n(\omega) \in G_i^n$, from the relations $(\omega, \arg_i^n(\omega)) \in P_{(i,s,k_n)}^2$, which hold (for every $\omega \in P_i$) for some $(s, k_n) \in S_i \times K_n$;
- * from Assertion (i) and Lemma 2-(v), the belief, $\pi_i^\eta := \frac{1}{1+\eta\#\underline{S}}(\pi_i + \eta \sum_{s \in \underline{S}} \delta_s)$, where δ_s is (for each $s \in \underline{S}$) the Dirac's measure of ω_s^η ;
- * the support of $\pi_i^\eta \in \mathcal{B}$, denoted by $P_i^\eta := P(\pi_i^\eta) = P_i \cup \{\omega_s^\eta\}_{s \in \underline{S}}$;
- * for all $(\omega := (s, p_s), z) \in \mathcal{M} \times \mathbb{R}^J$, the set $B_i(\omega, z) := \{x \in \mathbb{R}_+^L : p_s \cdot (x - e_{is}) \leq V(\omega) \cdot z\}$.

Then, the following Assertions hold, for each $i \in I$:

- (iv) $G_i^n \subset G_i^{n+1}$, $\forall n \in \mathbb{N}$, $\overline{G_i^\infty} = P_i$ and $\{\arg_i^n(\omega)\}_{n \in \mathbb{N}}$ converges to ω uniformly on P_i ;
- (v) $\forall s \in \underline{S}$, $\{x_{is}^\eta\} = \arg \max_{x \in B_i(\omega_s^\eta, z_i^\eta)} u_i(x_{i0}^\eta, x)$, along Assertion (ii); we let $x_{i\omega_s^\eta}^\eta := x_{is}^\eta$;
- (vi) the correspondence $\omega \in P_i^\eta \mapsto \arg \max_{x \in B_i(\omega, z_i^\eta)} u_i(x_{i0}^\eta, x)$ is a continuous mapping, denoted by $\omega \mapsto x_{i\omega}^\eta$. The mapping, $x_i^\eta : \omega \in \{0\} \cup P_i^\eta \mapsto x_{i\omega}^\eta$, defined from Assertions (ii) and (v) and above, is a consumption plan, henceforth referred to as $x_i^\eta \in X(\pi_i^\eta)$;
- (vii) $u_i^{\pi_i^\eta}(x_i^\eta) = \frac{1}{1+\eta\#\underline{S}} \lim_{n \rightarrow \infty} u_i^n(x_i^n) \in \mathbb{R}_+$.

Proof see the Appendix. □

5.3 An equilibrium of the initial economy

We now prove Assertion (ii) of Theorem 1, via the following Claim.

Claim 5 *The collection of prices, $(\omega_s^\eta) = \lim_{n \rightarrow \infty} (\omega_s^n)$, beliefs, (π_i^η) , allocation, (x_i^η) , and portfolios, $(z_i^\eta) = \lim_{n \rightarrow \infty} (z_i^n)$, of Lemma 3, is a C.F.E. of the economy \mathcal{E} .*

Proof Let $\mathcal{C}^\eta := ((\omega_s^\eta), (\pi_i^\eta), [(x_i^\eta, z_i^\eta)])$ be defined from Claim 5 and use the notations of Lemma 3. From Lemma 3-(ii)-(iii)-(v)-(vi), \mathcal{C}^η meets Conditions (c)-(d) of the above Definition 2 of equilibrium. From the definition, $\{\omega_s^\eta\}_{s \in \underline{S}} \subset \cap_{i=1}^m P(\pi_i^\eta)$, so \mathcal{C}^η meets

Condition (a) of Definition 2. To prove that \mathcal{C}^η is a C.F.E., it suffices to show that \mathcal{C}^η satisfies the relation $[(x_i^\eta, z_i^\eta)] \in \Pi_{i=1}^m B_i(\omega_0^\eta, \pi_i^\eta)$ and Condition (b) of Definition 2.

Let $i \in I$ be given. From the definition of \mathcal{C}^η , the relations $p_0^n \cdot (x_{i0}^n - e_{i0}) \leq -q^n \cdot z_i^n$ and $p_s^n \cdot (x_{is}^n - e_{is}) \leq V((s, p_s^{n-1})) \cdot z_i^n$ hold, for each $(n, s) \in \mathbb{N}_N \times \underline{\mathbf{S}}$, which yield in the limit (from the continuity of the scalar product): $p_0^\eta \cdot (x_{i0}^\eta - e_{i0}) \leq -q^\eta \cdot z_i^\eta$ and $p_s^\eta \cdot (x_{is}^\eta - e_{is}) \leq V(\omega_s^\eta) \cdot z_i^\eta$, for each $s \in \underline{\mathbf{S}}$. The relations $p_{s,\omega} \cdot (x_{i\omega}^\eta - e_{is}) \leq V(\omega) \cdot z_i^\eta$ hold, for all $(s, \omega := (s, p_s)) \in S_i \times P_i$, from Lemma 3-(v)-(vi). This implies, from Lemma 3-(vi): $[(x_i^\eta, z_i^\eta)] \in \Pi_{i=1}^m B_i(\omega_0^\eta, \pi_i^\eta)$.

Assume, by contraposition, that \mathcal{C}^η fails to meet Definition 2-(b), then, there exist $i \in I$, $(x, z) \in B_i(\omega_0^\eta, \pi_i^\eta)$ and $\varepsilon \in \mathbb{R}_{++}$, such that:

$$(I) \quad \varepsilon + u_i^{\pi_i^\eta}(x_i^\eta) < u_i^{\pi_i^\eta}(x).$$

We may assume that there exists $\delta \in \mathbb{R}_{++}$, such that:

$$(II) \quad x_\omega^l \geq \delta, \text{ for every } (\omega, l) \in \{0\} \cup P_i^\eta \times \mathcal{L}.$$

If not, for every $\alpha \in [0, 1]$, we define the strategy $(x^\alpha, z^\alpha) := ((1 - \alpha)x + \alpha e_i, (1 - \alpha)z)$, which belongs to $B_i(\omega_0^\eta, \pi_i^\eta)$, a convex set. From Assumption A1, the strategy (x^α, z^α) meets relations (II) whenever $\alpha > 0$. Moreover, from relation (I) and the uniform continuity of $(\alpha, \omega) \in [0, 1] \times P_i^\eta \mapsto u_i(x_\omega^\alpha, x_\omega^\alpha)$ on a compact set (which holds from Assumption A2 and the relation $x \in X(\pi_i^\eta)$), the strategy (x^α, z^α) also meets relation (I), for every $\alpha > 0$, small enough. So, we may assume relations (II).

We let the reader check, as immediate from the relations $(x, z) \in B_i(\omega_0^\eta, \pi_i^\eta)$ and $\pi_i^\eta \in \mathcal{B}$ (and the definition of a belief), from Lemma 3-(i), the relations (I) – (II), Assumption A2, and the same continuity arguments as above (and the continuity of the scalar product), that we may also assume there exists $\gamma \in \mathbb{R}_{++}$, such that:

$$(III) \quad p_0^\eta \cdot (x_0 - e_{i0}) \leq -\gamma - q^\eta \cdot z \text{ and } p_s \cdot (x_\omega - e_{is}) \leq -\gamma + V(\omega) \cdot z, \forall \omega := (s, p_s) \in P_i^\eta.$$

From (III), the continuity of the scalar product (hence, of $\omega \mapsto V(\omega)$) and Lemma 3-(i)-(iii)-(iv), there exists $N_1 \in \mathbb{N}_N$, such that, for every $n \geq N_1$:

$$(IV) \quad \begin{cases} p_0^n \cdot (x_0 - e_{i0}) \leq -q^n \cdot z \\ p_s^n \cdot (x_{\omega_s^\eta} - e_{is}) \leq V^n(s) \cdot z, \forall s \in \underline{\mathbf{S}} \\ p_s \cdot (x_\omega - e_{is}) \leq V(\omega) \cdot z, \forall \omega := (s, p_s) \in G_i^n \end{cases}.$$

Along relations (IV) and Lemma 3-(i)-(v)-(vi), for each $n \geq N_1$, we let $(x^n, z) \in B_i^n(S_i^n, (\omega_s^n))$ be the strategy defined by $x_0^n := x_0$, $x_s^n := x_{\omega_s^\eta}$, for every $s \in \underline{\mathbf{S}}$, and $x_{s^n}^n := x_\omega$, for every $s^n := (i, \omega) \in \tilde{S}_i^n$, and recall that:

- $u_i^{\pi_i^\eta}(x) := \frac{1}{1+\eta\#\underline{\mathbf{S}}} \int_{\omega \in P_i} u_i(x_0, x_\omega) du_i^{\pi_i}(\omega) + \frac{\eta}{1+\eta\#\underline{\mathbf{S}}} \sum_{s \in \underline{\mathbf{S}}} u_i(x_0, x_{\omega_s^\eta});$
- $u_i^n(x^n) := \sum_{s^n \in S_i^n \setminus \underline{\mathbf{S}}} u_i(x_0, x_{s^n}) \pi_i^n(s^n) + \eta \sum_{s \in \underline{\mathbf{S}}} u_i(x_0, x_{\omega_s^\eta}).$

Then, from above, Lemma 3-(i)-(iv) and the uniform continuity of $x \in X(\pi_i^\eta)$ and u_i on compact sets, there exists $N_2 \geq N_1$ such that (with $\beta := (1+\eta\#\underline{\mathbf{S}})$):

$$(V) \quad |\beta u_i^{\pi_i^\eta}(x) - u_i^n(x^n)| \leq \int_{\omega \in P_i} |u_i(x_0, x_\omega) - u_i(x_0, x_{\arg_i^n(\omega)})| du_i^{\pi_i}(\omega) < \frac{\varepsilon}{2}, \text{ for every } n \geq N_2.$$

From equilibrium conditions and Lemma 3-(vii), there exists $N_3 \geq N_2$, such that:

$$(VI) \quad u_i^n(x^n) \leq u_i^n(x_i^n) < \frac{\varepsilon}{2} + \beta u_i^{\pi_i^\eta}(x_i^\eta), \text{ for every } n \geq N_3 \text{ (with } \beta := (1+\eta\#\underline{\mathbf{S}})).$$

Let $n \geq N_3$ be given. The above Conditions (I)-(V)-(VI) yield, jointly:

$$\beta u_i^{\pi_i^\eta}(x) < \frac{\varepsilon}{2} + u_i^n(x^n) < \varepsilon + \beta u_i^{\pi_i^\eta}(x_i^\eta) < \beta u_i^{\pi_i^\eta}(x).$$

This contradiction proves that \mathcal{C}^η meets Condition (b) of Definition 2, hence, from above, that \mathcal{C}^η is a C.F.E. The sets Ω , of C.F.E., and Δ , of minimum uncertainty, of Definition 5, which contains $\{\omega_s^\eta\}_{s \in \underline{\mathbf{S}}}$, are non-empty, i.e., Theorem 1-(ii) holds. \square

Claim 6, below, completes the proof of Theorem 1 via the following Lemma.

Lemma 4 For each $(i, k) \in I \times \mathbb{N}$, we let $\eta_k := \frac{1}{2^k}$, denote simply $u_i^k := u_i^{\pi_i^{\eta_k}}$ and by $\mathcal{C}^k = ((\omega_s^k), (\pi_i^k), [(x_i^k, z_i^k)])$ the related C.F.E., \mathcal{C}^{η_k} , of Claim 5, and we define the set, $B_i(\omega, z) := \{x \in \mathbb{R}_+^L : p_s \cdot (x - e_{is}) \leq V(\omega) \cdot z\}$, for all $(\omega := (s, p_s), z) \in P_i \times \mathbb{R}^J$. Then, whenever $\Delta \subset \cap_{i=1}^m P_i$ along Definition 5, the following Assertions hold for each $i \in I$:

- (i) for each $s \in \underline{S}'$, it may be assumed to exist prices, $\omega_s^* = \lim_{k \rightarrow \infty} \omega_s^k \in \mathcal{M}_s$, such that $\{\omega_s^*\}_{s \in \underline{S}} \subset \cap_{i=1}^m P_i$, and consumptions, $x_{is}^* = \lim_{k \rightarrow \infty} x_{is}^k$, such that $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$;
- (ii) it may be assumed to exist portfolios, $z_i^* = \lim_{k \rightarrow \infty} z_i^k$, such that $\sum_{i=1}^m z_i^* = 0$;
- (iii) $\forall s \in \underline{S}$, $\{x_{is}^*\} = \arg \max_{x \in B_i(\omega_s^*, z_i^*)} u_i(x_{i0}^*, x)$ along Assertion (i); we let $x_{i\omega_s^*}^* := x_{is}^*$;
- (iv) the correspondence $\omega \in P_i \mapsto \arg \max_{x \in B_i(\omega, z_i^*)} u_i(x_{i0}^*, x)$ is a continuous mapping, denoted by $\omega \mapsto x_{i\omega}^*$. The mapping $x_i^* : \omega \in \{0\} \cup P_i \mapsto x_{i\omega}^*$, defined from Assertions (i)-(iii) and above, is a consumption plan, referred to as $x_i^* \in X(\pi_i)$;
- (v) for every $x \in X(\pi_i)$, $u_i^{\pi_i}(x) = \lim_{k \rightarrow \infty} u_i^k(x) \in \mathbb{R}_+$ and $u_i^{\pi_i}(x_i^*) = \lim_{k \rightarrow \infty} u_i^k(x_i^k) \in \mathbb{R}_+$.

Proof see the Appendix. □

Claim 6 Whenever $\Delta \subset \cap_{i=1}^m P_i$, the collection of prices, $(\omega_s^*) = \lim_{k \rightarrow \infty} (\omega_s^k)$, beliefs, (π_i) , allocation, (x_i^*) , and portfolios, $(z_i^*) = \lim_{k \rightarrow \infty} (z_i^k)$, of Lemma 4, is a C.F.E.

Proof The proof is similar to that of Claim 5. We assume that $\Delta \subset \cap_{i=1}^m P_i$ and let $\mathcal{C}^* := ((\omega_s^*), (\pi_i), [(x_i^*, z_i^*)])$ be defined from Lemma 4, whose notations will be used throughout. Given $(i, k) \in I \times \mathbb{N}$, the relations $\{\omega_s^k\}_{s \in \underline{S}} \subset \Delta \subset \cap_{i=1}^m P_i$ hold from Claim 5, and imply that $P(\pi_i^k) = P_i$, hence that, $B_i(\omega_0^k, \pi_i^k)$ and $B_i(\omega_0^k, \pi_i^k)$ may only differ by one budget constraint at $t = 0$. From Lemma 4, \mathcal{C}^* meets Conditions (a)-(c)-(d) of Definition 2. Moreover, for every $(i, k) \in I \times \mathbb{N}$, the relations $p_0^k \cdot (x_{i0}^k - e_{i0}) \leq -q^k \cdot z_i^k$ hold, from Claim 5, and, passing to the limit, yield $p_0^* \cdot (x_{i0}^* - e_{i0}) \leq -q^* \cdot z_i^*$, which implies, from Lemma 4-(iv) and above: $(x_i^*, z_i^*) \in B_i(\omega_0^*, \pi_i)$, for each $i \in I$. Thus, Claim 6 will

be proved if we show that \mathcal{C}^* meets Condition (b) of Definition 2. By contraposition, assume this is not the case, i.e., there exists $(i, (x, z), \varepsilon) \in I \times B_i(\omega_0^*, \pi_i) \times \mathbb{R}_{++}$, such that:

$$(I) \quad \varepsilon + u_i^{\pi_i}(x_i^*) < u_i^{\pi_i}(x).$$

By the replacing (if necessary) the better strategy, $(x, z) \in B_i(\omega_0^*, \pi_i)$, by the strategy $[(1 - \eta)(x, z) + \eta(e_i, 0)] \in B_i(\omega_0^*, \pi_i)$, for $\eta > 0$ small enough, we may always assume (from Assumptions A1-A2) that the first period consumption, x_0 , is positive. Then, from the relation $(x, z) \in B_i(\omega_0^*, \pi_i)$ and Assumptions A1-A2, the relation:

$$(II) \quad p_0^* \cdot (x_0 - e_{i0}) \leq -\gamma - q^* \cdot z, \text{ for some } \gamma \in \mathbb{R}_{++}, \text{ may also be assumed.}$$

From (II), Lemma 4-(i), continuity arguments and the identity of $B_i(\omega_0^*, \pi_i)$ and $B_i(\omega_0^k, \pi_i^k)$ on all second period budget constraints, there exists $K \in \mathbb{N}$, such that:

$$(III) \quad (x, z) \in B_i(\omega_0^k, \pi_i) = B_i(\omega_0^k, \pi_i^k), \text{ for every } k \geq K.$$

Relations (I)-(III), Lemma 4-(v) and the fact that \mathcal{C}^k is a C.F.E., yield:

$$(IV) \quad u_i^{\pi_i}(x) < \frac{\varepsilon}{2} + u_i^k(x) \leq \frac{\varepsilon}{2} + u_i^k(x_i^k) < \varepsilon + u_i^{\pi_i}(x_i^*) < u_i^{\pi_i}(x), \text{ for } k \geq K \text{ big enough.}$$

This contradiction proves that \mathcal{C}^* meets Definition 2-(b), hence, is a C.F.E.

The proof of Theorem 1 is now complete. □

Appendix: proof of the Lemmas

Lemma 1 *There exists $N \in \mathbb{N}$, such that the following Assertion holds:*

$$(i) \quad \forall (P_i^*) \leq (P_i), (G_i^N \subset P_i^*, \forall i \in I) \Rightarrow (Q_c[(P_i^*)] \neq \emptyset).$$

Proof Let the arbitrage-free anticipation structure, $(P_i) \in \mathcal{AS}$, and sequences, $\{(G_i^n)\}_{n \in \mathbb{N}}$, be defined as in Section 5. For each $(i, n) \in I \times \mathbb{N}$, we consider the vector

space $Z_i^n := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in G_i^n\}$ and its orthogonal, $Z_i^{n\perp}$, and, similarly, $Z_i^* := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in P_i\}$ and $Z_i^{*\perp}$. We show, first, that, for each $i \in I$, there exists $N_i \in \mathbb{N}$, such that $Z_i^n = Z_i^*$, for every $n \geq N_i$. Indeed, since $\{G_i^n\}_{n \in \mathbb{N}}$ is non-decreasing, $\{Z_i^n\}_{n \in \mathbb{N}}$ is non increasing in \mathbb{R}^J , hence, stationary, that is, there exists $N_i \in \mathbb{N}$, such that $Z_i^n = Z_i^{N_i}$, for every $n \geq N_i$. From the definition, $Z_i^* \subset Z_i^{N_i}$. From the fact that $\lim_{n \rightarrow \infty} \nearrow G_i^n = \cup_{n \in \mathbb{N}} G_i^n$ is dense in P_i , we easily show, by contraposition, that $Z_i^* = Z_i^{N_i}$ (for all $n > N_i$, take $z_n \in Z_i^{*\perp} \cap Z_i^n$, such that $\|z_n\| = 1$ and derive a contradiction). We let $N^o = \max_{i \in I} N_i$ and define the compact set, $Z := \{(z_i) \in \prod_{i=1}^m Z_i^{*\perp} : \|(z_i)\| = 1, \sum_{i=1}^m z_i \in \sum_{i=1}^m Z_i^*\}$.

Assume, by contraposition, that Lemma 2 fails. Then, from Claim 1-(i) and above, for every $n \geq N^o$, there exist an integer, $N_n \geq n$, expectation sets, $(P_i^{N_n})$, such that $G_i^{N_n} \subset P_i^{N_n} \subset P_i$, for each $i \in I$, and portfolios, $(z_i^n) \in Z$, such that $V(\omega_i) \cdot z_i^n \geq 0$ holds for every $(i, \omega_i) \in I \times P_i^{N_n}$, with one strict inequality. The sequence, $\{(z_i^n)\}_{n \geq N^o}$, may be assumed to converge in a compact set, say to $(z_i^*) \in Z$. From the continuity of the scalar product and the fact that, for each $i \in I$, $\lim_{n \rightarrow \infty} G_i^n = \cup_{n \in \mathbb{N}} G_i^n$ is dense in P_i , the above relations on $\{(z_i^n)\}_{n \geq N^o}$, imply, in the limit, that $V(\omega_i) \cdot z_i^* \geq 0$ holds, for every $(i, \omega_i) \in I \times P_i$, with one strict inequality, since $(z_i^*) \in Z$. This contradicts the fact that (P_i) is arbitrage-free. This contradiction proves Lemma 1. \square

Lemma 2 *The generic (n, η) -economy admits an equilibrium, namely a collection of prices, $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$, at $t = 0$, and $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$, in each state $s \in \underline{\mathbf{S}}$, and strategies, $(x_i^n, z_i^n) \in B_i^n(S_i^n, (\omega_s^n))$, defined for each $i \in I$, such that:*

- (i) $\forall i \in I, (x_i^n, z_i^n) \in \arg \max_{(x, z) \in B_i^n(S_i^n, (\omega_s^n))} u_i^n(x);$
- (ii) $\forall s \in \underline{\mathbf{S}}', \sum_{i=1}^m (x_{is}^n - e_{is}) = 0;$
- (iii) $\sum_{i=1}^m z_i^n = 0.$

Moreover, the equilibrium prices and allocations satisfy the following Assertions:

(iv) $\forall (n, i, s) \in \mathbb{N}_N \times I \times \underline{\mathbf{S}}', x_{is}^n \in [0, e]^L$, where $e := \max_{(s,l) \in S' \times \mathcal{L}} \sum_{i=1}^m e_{is}^l$;

(v) $\exists \varepsilon \in]0, 1] : p_s^{nl} \geq \varepsilon, \forall (n, s, l) \in \mathbb{N}_N \times \underline{\mathbf{S}} \times \mathcal{L}$.

Proof We recall that $\mathbb{N} := \mathbb{N} \setminus \{1, \dots, N-1\}$, along Lemma 1, and let $n \in \mathbb{N}_N$ be given. From Lemma 1 and the fact that $\underline{\mathbf{S}}$ is a set of common states for all agents, the payoff and information structure, $[V^n, (S_i^n)]$, is arbitrage-free, along [6], Definition 4, p. 260, on a purely financial market. Moreover, the (n, η) -economy is, formally, one of the type presented in [6] and, from above, admits an equilibrium along Definition 3 and Theorem 1 of [6] and its proof, more precisely (up to slightly changed notations), it admits a collection of prices, $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$, and $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$, for each $s \in \underline{\mathbf{S}}$, and strategies, $(x_i^n, z_i^n) \in B_i^n(S_i^n, (\omega_s^n))$, defined for each $i \in I$, which satisfy Assertions (i)-(ii)-(iii) of Lemma 2 (which, hence, hold). The proof of Assertions (iv)-(v) is similar (simpler) to that above of Theorem 1-(i), hence, left to readers. \square

Lemma 3 *For the above sequence, $\{\mathcal{C}^n\}$, of equilibria, it may be assumed to exist:*

(i) $\omega_0^\eta := (p_0^\eta, q^\eta) = \lim_{n \rightarrow \infty} \omega_0^n \in \mathcal{M}_0$ and $\omega_s^\eta := (s, p_s^\eta) = \lim_{n \rightarrow \infty} \omega_s^n \in \mathcal{M}_s$, for each $s \in \underline{\mathbf{S}}$;

(ii) $(x_{is}^\eta) := \lim_{n \rightarrow \infty} (x_{is}^n)_{i \in I} \in \mathbb{R}^{Lm}$, such that $\sum_{i \in I} (x_{is}^\eta - e_{is}) = 0$, for each $s \in \underline{\mathbf{S}}'$;

(iii) $(z_i^\eta) = \lim_{n \rightarrow \infty} (z_i^n)_{i \in I} \in \mathbb{R}^{Jm}$, such that $\sum_{i=1}^m z_i^\eta = 0$.

Moreover, we define, for each $i \in I$, the following sets and mappings:

* $G_i^\infty := \cup_{n \in \mathbb{N}} G_i^n = \lim_{n \rightarrow \infty} \nearrow G_i^n \subset P_i$;

* for each $n \in \mathbb{N}$, the mapping, $\omega \in P_i \mapsto \arg_i^n(\omega) \in G_i^n$, from the relations

$(\omega, \arg_i^n(\omega)) \in P_{(i,s,k_n)}^2$, which hold (for every $\omega \in P_i$) for some $(s, k_n) \in S_i \times K_n$;

* from Assertion (i) and Lemma 2-(v), the belief, $\pi_i^\eta := \frac{1}{1+\eta\#\underline{\mathbf{S}}}(\pi_i + \eta \sum_{s \in \underline{\mathbf{S}}} \delta_s)$,

where δ_s is (for each $s \in \underline{\mathbf{S}}$) the Dirac's measure of ω_s^η ;

* the support of $\pi_i^\eta \in \mathcal{B}$, denoted by $P_i^\eta := P(\pi_i^\eta) = P_i \cup \{\omega_s^\eta\}_{s \in \underline{\mathbf{S}}}$;

* for all $(\omega := (s, p_s), z) \in \mathcal{M} \times \mathbb{R}^J$, the set $B_i(\omega, z) := \{x \in \mathbb{R}_+^L : p_s \cdot (x - e_{is}) \leq V(\omega) \cdot z\}$.

Then, the following Assertions hold, for each $i \in I$:

- (iv) $G_i^n \subset G_i^{n+1}$, $\forall n \in \mathbb{N}$, $\overline{G_i^\infty} = P_i$ and $\{\arg_i^n(\omega)\}_{n \in \mathbb{N}}$ converges to ω uniformly on P_i ;
- (v) $\forall s \in \underline{\mathbf{S}}$, $\{x_{is}^\eta\} = \arg \max_{x \in B_i(\omega_s^\eta, z_i^\eta)} u_i(x_{i0}^\eta, x)$, along Assertion (ii); we let $x_{i\omega_s^\eta}^\eta := x_{is}^\eta$;
- (vi) the correspondence $\omega \in P_i^\eta \mapsto \arg \max_{x \in B_i(\omega, z_i^\eta)} u_i(x_{i0}^\eta, x)$ is a continuous mapping, denoted by $\omega \mapsto x_{i\omega}^\eta$. The mapping, $x_i^\eta : \omega \in \{0\} \cup P_i^\eta \mapsto x_{i\omega}^\eta$, defined from Assertions (ii) and (v) and above, is a consumption plan, henceforth referred to as $x_i^\eta \in X(\pi_i^\eta)$;
- (vii) $u_i^{\pi_i^\eta}(x_i^\eta) = \frac{1}{1+\eta\#\underline{\mathbf{S}}} \lim_{n \rightarrow \infty} u_i^n(x_i^n) \in \mathbb{R}_+$.

Proof Assertions (i)-(ii) result from Lemma 2-(iv) and compactness arguments. \square

Assertion (iii) For all $(i, n) \in I \times \mathbb{N}_N$, we let $Z_i^* := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in P_i\}$ and recall from the proof of Lemma 1 that $Z_i^* = \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in G_i^n\}$. We show that the portfolio sequence $\{(z_i^n)_{i \in I}\}$ is bounded in \mathbb{R}^{Jm} . Indeed, let $\delta := \max_{i \in I} \|e_i\|$. The definition of $\{\mathcal{C}^n\}_{n \in \mathbb{N}_N}$ yields, from budget constraints and clearance conditions:

$$(I) \quad \left[\sum_{i=1}^m z_i^n = 0 \text{ and } V(\omega_i) \cdot z_i^n \geq -\delta, \forall (i, \omega_i) \in I \times G_i^n \right], \text{ for every } n \in \mathbb{N}_N.$$

Assume, by contradiction, $\{(z_i^n)\}$ is unbounded, i.e., there exists an extracted sequence, $\{(z_i^{\varphi(n)})\}$, such that $n < \|(z_i^{\varphi(n)})\| \leq n+1$, for all $n \in \mathbb{N}_N$. From (I), the portfolios $(z_i^{*n}) := \frac{1}{n}(z_i^{\varphi(n)})$ meet, for all $n \in \mathbb{N}_N$, the relations $1 < \|(z_i^{*n})\| \leq 1 + \frac{1}{n}$ and:

$$(II) \quad \sum_{i=1}^m z_i^n = 0 \text{ and } V(\omega_i) \cdot z_i^{*n} \geq -\frac{\delta}{n}, \forall (i, \omega_i) \in I \times G_i^n.$$

From (II), the density of G_i^∞ in P_i , scalar product continuity and above, the sequence $\{(z_i^{*n})\}$ may be assumed to converge, say to (\bar{z}_i) , such that $\|(\bar{z}_i)\| = 1$ and:

$$(III) \quad \sum_{i=1}^m \bar{z}_i = 0 \text{ and } V(\omega_i) \cdot \bar{z}_i \geq 0, \forall (i, \omega_i) \in I \times P_i.$$

Relations (III) and the fact that (P_i) is arbitrage-free imply, from Claim 1, $(\bar{z}_i) \in \Pi_{i=1}^m Z_i^*$ and, from the elimination of useless deals (see sub-Section 2.4), $(\bar{z}_i) = 0$, which contradicts the fact that $\|(\bar{z}_i)\| = 1$. Hence, the sequence $\{(z_i^n)\}$ is bounded

and may be assumed to converge, say to $(z_i^\eta) \in \mathbb{R}^{Jm}$. Then, the relation $\sum_{i=1}^m z_i^\eta = 0$ results asymptotically from the clearance conditions of Lemma 2-(iv). \square

Assertions (iv) is immediate from the definitions and compactness arguments. \square

Assertion (v) Let $(i, s) \in I \times \underline{\mathbf{S}}$ be given. For every $(n, \omega := (s, p_s), \omega', z) \in \mathbb{N}_N \times \mathcal{M}_s \times \mathcal{M}_s \times \mathbb{R}^J$, we consider the following (possibly empty) sets:

$$B_i(\omega, z) := \{y \in \mathbb{R}_+^L : p_s \cdot (y - e_{is}) \leq V(\omega) \cdot z\} \text{ and } B'_i(\omega, \omega', z) := \{y \in \mathbb{R}_+^L : p_s \cdot (y - e_{is}) \leq V(\omega') \cdot z\}.$$

For each $n > N$, the fact that \mathcal{C}^n is a (n, η) -equilibrium implies, from Lemma 2:

$$(I) \quad (\omega_s^{n-1}, \omega_s^n) \in \mathcal{M}_s^2 \text{ and } x_{is}^n \in \arg \max_{y \in B'_i(\omega_s^n, \omega_s^{n-1}, z_i^n)} u_i(x_{i0}^n, y).$$

As a standard application of Berge's Theorem (see, e.g., [8], p. 19), the correspondence $(x, \omega, \omega', z) \in \mathbb{R}_+^L \times \mathcal{M}_s \times \mathcal{M}_s \times \mathbb{R}^J \mapsto \arg \max_{y \in B'_i(\omega, \omega', z)} u_i(x, y)$, which is actually a mapping (from Assumption A2), is continuous at $(x_{i0}^\eta, \omega_s^\eta, \omega_s^\eta, z_i^\eta)$, since u_i and B'_i are continuous. Moreover, the relation $(x_{i0}^\eta, x_{is}^\eta, \omega_s^\eta, z_i^\eta) = \lim_{n \rightarrow \infty} (x_{i0}^n, x_{is}^n, \omega_s^n, z_i^n)$ holds from Lemma 2-(i)-(ii)-(iii). Hence, the relations (I) pass to that limit and yield:

$$\{x_{i\omega_s^\eta}^\eta\} := \{x_{is}^\eta\} = \arg \max_{y \in B_i(\omega_s^\eta, z_i^\eta)} u_i(x_{i0}^\eta, y). \quad \square$$

Assertion (vi) Let $i \in I$ be given. For every $(\omega, n) \in P_i \times \mathbb{N}_N$, the fact that \mathcal{C}^n is a (n, η) -equilibrium and Assumption A2 yield:

$$(I) \quad \{x_{i\arg_i^n(\omega)}^n\} = \arg \max_{y \in B_i(\arg_i^n(\omega), z_i^n)} u_i(x_{i0}^n, y).$$

From Lemma 2-(ii)-(iii)-(iv), the relation $(\omega, x_{i0}^\eta, z_i^\eta) = \lim_{n \rightarrow \infty} (\arg_i^n(\omega), x_{i0}^n, z_i^n)$ holds, whereas, from Assumption A2 and ([8], p. 19), the correspondence $(x, \omega, z) \in \mathbb{R}_+^L \times P_i \times \mathbb{R}^J \mapsto \arg \max_{y \in B_i(\omega, z)} u_i(x, y)$ is a continuous mapping, since u_i and B_i are

continuous. Hence, passing to the limit into relations (I) yields a continuous mapping, $\omega \in P_i \mapsto x_{i\omega} := \arg \max_{y \in B_i(\omega, z_i^\eta)} u_i(x_{i0}^\eta, y)$, which, from Lemma 3-(v) and above, is embedded into a continuous mapping, $x_i^\eta : \omega \in \{0\} \cup P_i^\eta \mapsto x_{i\omega}^\eta$, i.e., $x_i^\eta \in X(\pi_i^\eta)$. \square

Assertion (vii) Let $i \in I$ and $x_i^\eta \in X(\pi_i^\eta)$ be given, along Lemma 3-(vi). Let $\varphi_i : (x, \omega, z) \in \mathbb{R}_+^L \times P_i \times \mathbb{R}^J \mapsto \arg \max_{y \in B_i(\omega, z)} u_i(x, y)$ be defined on its domain. By the same token as for proving Assertion (vi), φ_i and $U_i : (x, \omega, z) \in \mathbb{R}_+^L \times P_i \times \mathbb{R}^J \mapsto u_i(x, \varphi_i(x, \omega, z))$ are continuous mappings and, moreover, the relations $u_i(x_{i0}^\eta, x_{i\omega}^\eta) = U_i(x_{i0}^\eta, \omega, z_i^\eta)$ and $u_i(x_{i0}^n, x_{i \arg_i^n(\omega)}^n) = U_i(x_{i0}^n, \arg_i^n(\omega), z_i^n)$ hold, for every $(\omega, n) \in P_i \times \mathbb{N}_N$. Then, the uniform continuity of u_i and U_i on compact sets, and Lemma 3-(ii)-(iii)-(iv) yield:

$$(I) \quad \forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}_N : \forall n > N_\varepsilon, \forall \omega \in P_i, \\ |u_i(x_{i0}^\eta, x_{i\omega}^\eta) - u_i(x_{i0}^n, x_{i \arg_i^n(\omega)}^n)| + \sum_{s \in \underline{\mathbf{S}}} |u_i(x_{i0}^\eta, x_{i\omega_s^\eta}^\eta) - u_i(x_{i0}^n, x_{is}^n)| < \varepsilon.$$

Moreover, with $\beta := (1 + \eta \# \underline{\mathbf{S}})$, we recall the following definitions, for every $n \in \mathbb{N}_N$:

$$(II) \quad \beta u_i^{\pi_i^\eta}(x_i^\eta) := \int_{\omega \in P_i} u_i(x_{i0}^\eta, x_{i\omega}^\eta) d\pi_i(\omega) + \eta \sum_{s \in \underline{\mathbf{S}}} u_i(x_{i0}^\eta, x_{is}^\eta); \\ (III) \quad u_i^n(x_i^n) := \int_{\omega \in P_i} u_i(x_{i0}^n, x_{i \arg_i^n(\omega)}^n) d\pi_i(\omega) + \eta \sum_{s \in \underline{\mathbf{S}}} u_i(x_{i0}^n, x_{is}^n).$$

Then, Lemma 3-(vii) results immediately from relations (I)-(II)-(III) above. \square

Proof of Lemma 4 It is similar to that of Lemma 3, hence, left to the reader. \square

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